Performance of classical control chart procedures in the presence of VAR(1) process

A.A. Kalgonda
Dept. of Statistics, The New College, Kolhapur-416012, India
annagmk@rediffmail.com; +91 9922770979

Abstract

When the data are autocorrelated, the usual Shewhart $\bar{X}$, EWMA, and CUSUM control chart limits obtained (under the independence assumptions) are obviously not appropriate for monitoring a process. In the sense, these charts increase the false alarm rate resulting into the decrease of ARL. In multivariate case, when the data are autocorrelated, the problems become more complex. In this study, the performance of using classical multivariate control chart procedures in the presence of autocorrelation on stationary VAR(1) process is discussed and illustrated with the help of an example.

Keywords: Multivariate control chart procedures, VAR(1) process, false alarm rate, autocorrelation.

Introduction

In constructing control charts such as the Shewhart $\bar{X}$, CUSUM, it is frequently assumed that the data generated from the in control process are independent and identically distributed. Further, whenever the process goes out of control, one proceeds to search for the factors responsible in terms of assignable or special causes. However, as explicitly pointed out by Alwan and Roberts (1995), in reality, most of the times, the assumption of independence is violated. Consequently, it becomes difficult either to detect the status of the process correctly or identify the factors responsible for an out of control state due to the presence of 'systematic non-random pattern' (called common cause) or autocorrelation in the data. Examples of this phenomenon are numerous. For instance, many chemical processes due to wear on equipment, environmental and chemical contamination are prone to generate autocorrelated data (Mason and Young, 2002).

Alwan and Roberts (1995) observed that in 235 control charts applications published in the literature, about 85% displaced incorrect control limits. More than half of these displacements of control limits were due to violation of the independence assumption. Two general approaches are suggested to overcome the problem of autocorrelation in the univariate setup. The first approach is to adjust or modify the standard control limits to account for the autocorrelation (Vasilopoulos and Stamboulis, 1978; Zhang, 1998). The second approach is to fit an appropriate time series model to the data and then construct a control chart based on the residuals (Alwan and Roberts, 1988; Montgomery and Mastrandelo, 1991; Wardell et al., 1994; Lu and Reynolds, 1999). Each approach has its own limitations. For instance, the first approach needs parameter estimation, whereas in the second approach, the major limitation is to find a suitable time series model before obtaining the residuals.

Consequently, if the fitted model is not adequate, the residual based charts may not serve the purpose. In multivariate case, when the data are autocorrelated, the problem becomes even more complex. In multivariate autocorrelated process notable work are of Theodossion (1993), Kramer and Schmid (1997), Kalgonda and Kulkarni (2004). In this study, emphasis is given on the effect of performance of classical multivariate control chart procedure in the presence of autocorrelation.

Materials and methods

Vector autoregressive model

Let $X_t$ denote an observation vector of order $p \times 1$ generated by a process at time $t$. Traditionally, it is assumed that if the process is in control state, it generates observations with a constant mean vector $\mu$ and covariance matrix $\Sigma$. As done in a classical univariate case, this situation can be represented in the following model form

$$X_t = \mu + \xi_t$$  \hspace{1cm} (1)

Where, $\xi_t$ is a vector of independent errors.

However, when the assumption of independence of observation vectors is violated, it affects the control limits, thereby increasing the number of false alarms. To overcome such a problem, a better approach is to model the patterns by a suitable time series model by taking into account the structure of the dependence. This is usually done through using autoregressive (AR) models of order $q$ and in a multivariate case, a vector autoregressive (VAR) model. In what follows, I will restrict to $q = 1$, as this is frequently applicable model in practice. The following discussion is based on Reinsel (1993).
In univariate case, a first order autoregressive (AR(1)) model is
\[ x_t = \mu_t + \Phi(x_{t-1} - \mu_{t-1}) + \varepsilon_t, \quad t = 1, 2, \ldots \] (2)

Where \( \mu_t \) and \( \mu_{t-1} \) are the means at time \( t \) and \( t-1 \) respectively, \( \Phi \) is the autoregressive coefficient, and \( x_{t-1} \) is an observation at time \( t-1 \). Such models are commonly used in practice to handle a wide range of autocorrelation existing in the process data (Lu and Reynolds, 1999; Montgomery and Mastrangelo, 1991).

Multivariate version of AR (1) is called vector autoregressive model of order one (VAR(1)). Kramer and Schmid (1997) have used VAR(1) model in the modification of multivariate EWMA charts in the presence of autocorrelation. Let \( \{ X_t, t = 1, 2, \ldots \} \) be a vector autoregressive process.
\[ X_t = \mu_t + \Phi(x_{t-1} - \mu_{t-1}) + \varepsilon_t, \quad t = 1, 2, \ldots \] (3)

Where \( \mu_t \) and \( \mu_{t-1} \) are the mean vectors at time \( t \) and \( t-1 \) respectively, \( \Phi \) is \( pxp \) matrix of autoregressive coefficients. Further, \( \varepsilon_t \)'s are vector of errors, which are independent and identically distributed with \( E(\varepsilon_t) = 0 \) and
\[ Cov(\varepsilon_t, \varepsilon_{t+k}) = \begin{cases} \sum & \text{for } k = 0 \\ 0 & \text{otherwise} \end{cases} \] (4)

The process \( \{ X_t \} \) can either be stationary or non-stationary. In process monitoring environment, the target values of mean vector of quality characteristics under consideration are required to set at target. For this, some type of regular adjustment to the process is required when \( \{ X_t \} \) is non-stationary and control charts can be used to monitor adjusted process. However, in this study, I restrict myself to a case where the process is stationary. Below, I present the conditions for stationarity of the process.

**Stationarity condition for VAR(1) process**

A vector process \( \{ X_t \} \) is stationary if the probability distribution of observations from the process is invariant with respect to shift in time (Reinsel, 1993). The stationarity ensures constancy of the means, variances and covariance through time. Precisely, for this reason in most of the multivariate cases, stationary processes are assumed.

For stationary process \( \{ X_t \} \),
\[ X_t = \mu + \Phi(x_{t-1} - \mu) + \varepsilon_t, \quad t = 1, 2, \ldots \] (6)

Further, for VAR (1) model, the stationarity condition needs that all eigenvalues of \( \Phi \), that is, the roots \( |\lambda I - \Phi| = 0 \), be less than one in absolute value. The VAR (1) process defined in (6) mainly differs from the classical model defined in (1) with respect to the parameter matrix \( \Phi \). If \( \Phi \) is a null matrix, the model (6) reduces to the model (1). Obviously, the covariance matrix for a process exhibiting VAR(1) model depends upon the autocorrelation parameter matrix \( \Phi \) and it is called the cross-covariance matrix.

**Results and discussion**

**Effect of autocorrelation on Hayter and Tsui’s chart**

In univariate case, when the process observations are autocorrelated, it has profound effect on the performance of the control charts. The average run length (ARL) is much shorter than is expected when the process is in control (Montgomery and Mastrangelo, 1991; Alwan, 1992; Wardell et al., 1994). As pointed out by Kramer and Schmid (1997), Lowry and Montgomery (1995) among others, this problem also extends to multivariate cases. Consequently, unnecessary out of control signals, when process is on target, may mislead the user and may cause unnecessary corrections to the process. To what extent this may happen is illustrated below. For this purpose, the multivariate control chart procedure suggested by Hayter and Tsui (1994) for independent observations is used.

Consider a bivariate vector \( X_t = (x_{1t}, x_{2t}) \) of observations from an autocorrelated stationary VAR(1) process at time \( t \) with the following parameters.
\[
\mu_0 = (\mu_0^1, \mu_0^2)', \quad \Phi = diag(a, b) \quad \text{and} \quad \Sigma = (\sigma_{ij}).
\]

Additionally, for the stationarity of the process, we assume that \( |a|, |b| < 1 \).

Suppose an operator or process engineer might not be aware of the problems that are caused by autocorrelation of the data, and consequently applies the Hayter and Tsui’s (1994) control chart based on their M-statistic as if the data were serially independent. In this case, the control statistic \( M \) becomes
\[
M = \max \left[ \frac{|x_1 - \mu_1|}{\sqrt{\sigma_{11}}}, \frac{|x_2 - \mu_2|}{\sqrt{\sigma_{22}}} \right] = \max \left[ M_1, M_2 \right]
\] (1)
Table 1. In control ARL and $\alpha$

<table>
<thead>
<tr>
<th></th>
<th>a</th>
<th>0.5</th>
<th>0.5</th>
<th>0.7</th>
<th>0.2</th>
<th>0.9</th>
<th>0.9</th>
</tr>
</thead>
<tbody>
<tr>
<td>b</td>
<td>0.5</td>
<td>0.5</td>
<td>0.7</td>
<td>0.2</td>
<td>0.9</td>
<td>0.2</td>
<td>0.9</td>
</tr>
<tr>
<td>ARL</td>
<td>200</td>
<td>96</td>
<td>197</td>
<td>182</td>
<td>5.086</td>
<td>5.425</td>
<td>3.18</td>
</tr>
<tr>
<td>Estimated $\alpha$</td>
<td>0.005</td>
<td>0.010</td>
<td>0.039</td>
<td>0.036</td>
<td>0.197</td>
<td>0.184</td>
<td>0.314</td>
</tr>
</tbody>
</table>

Where $M_i = \frac{|x_i - \mu_i|}{\sqrt{\sigma_{ii}}}$; $i = 1, 2$.

The corresponding critical value used for comparing $M$ is $C_{R, \alpha}$ where $R$ is the correlation matrix of the process and $\alpha$ is the probability of type I error.

Specifically to demonstrate the effect of autocorrelation on the performance of $M$ statistic, we select VAR (1) model as specified in Eq. (6). The values of $\Phi, \mu_0$ and $\Sigma$ as follows:

$\Phi = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}, \mu_0 = (0, 0)$, and $\Sigma = \begin{bmatrix} 1 & 0.5 \\ 0.5 & 1 \end{bmatrix}$

Where the values of a and b are shown in Table 1. Each time 10000 observations were generated inducing changes in $\Phi$ without changing $\mu_0$. For each observation, $M$ statistic was compared with $UCL = C_{R, 0.005} = 2.96$ at an error rate $\alpha = 0.005$.

The value of $C_{R, 0.005}$ is evaluated using simulation method (Hayter and Tsui, 1994) by constructing the empirical cumulative distribution function from 10,000 observations. The performance of $M$ statistic for change in $\Phi$ is judged with the help of computing the values of ARL and these are shown in Table 1.

Conclusion
The following are the conclusive points derived at the end of the study:
1. Even though the VAR(1) process is in control, the false alarm rate increases as the elements of parameter matrix $\Phi$ increases.
2. The severity of the use of $M$-statistic when the data is autocorrelated depends upon autocorrelation parameter matrix $\Phi$. It is evident from the false alarm rate $\alpha = 0.314$ corresponding to $a = b = 0.9$.

Thus, when the data are autocorrelated, it is essential to modify appropriately the control limits.

References